JOURNAL OF APPROXIMATION THEORY **46**, 119–128 (1986)

On the Work of G. Freud in the Theory of Interpolation of Functions

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Communicated by Paul G. Nevai

Received April 4, 1984; revised December 10, 1984

DEDICATED TO THE MEMORY OF GÉZA FREUD

The theory of interpolation of functions has been traditionally a favorite subject in Hungary. Starting with the famous result of L. Fejér, many Hungarian mathematicians contributed to this fruitful branch of approximation theory: E. Feldheim, G. Grünwald, P. Erdös, P. Turán, L. Kalmár, P. Szász, and their disciples from the younger generation. Although only about ten percent of his papers are devoted to interpolation theory, Géza Freud made significant progress in this field.

LAGRANGE INTERPOLATION

Let

$$x_{1n} < x_{2n} < \cdots < x_{nn} \qquad (n = 1, 2, \ldots) \tag{1}$$

be the nodes of interpolation,

$$l_{kn}(x) = \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{x - x_{in}}{x_{kn} - x_{in}}$$

the corresponding fundamental polynomials, and

$$\Lambda_n(x) = \sum_{k=1}^{n} |l_{kn}(x)|$$

the Lebesgue-function of interpolation. A classical result of Szegö states

0021-9045/86 $3.00

that if the nodes (1) are the roots of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ $(\alpha, \beta > -1)$, then

$$\Lambda_n(x) = O(\log n) \tag{2}$$

uniformly in every closed subinterval of $(-1, 1)$. Freud [1] proved that (2) remains true if (1) are the roots of the polynomials $P_n(x)$ orthonormal with respect to the weight function $w(x)$ satisfying

$$0 < m \leqslant w(x) \leqslant M \tag{3}$$

in closed subintervals of $(-1, 1)$, provided

$$|P_n(x)| \leqslant K \qquad (n = 1, 2, \ldots). \tag{4}$$

This generalization of Szegõ's result has a consequence for the convergence of the corresponding interpolating polynomials. Freud also proved [3] that under the conditions (3)–(4), for the interpolating polynomials

$$L_n(f, x) \stackrel{\text{def}}{=} \sum_{k=1}^{n} f(x_{kn})\, l_{kn}(x)$$

one has

$$\lim_{n \to \infty} L_n^{(m)}(f, x) = f^{(m)}(x) \tag{5}$$

uniformly in every closed subinterval $[a, b] \subset (-1, 1)$ provided

$$|f^{(m)}(x_1) - f^{(m)}(x_2)| = o\left(\log^{-1} \frac{1}{|x_1 - x_2|}\right) \qquad (a \leqslant x_1, x_2 \leqslant b). \tag{6}$$

If the weight function $w(x)$ satisfies only the lower restriction in (3) then (5) holds only if $f^{(m)}(x)$ satisfies the stronger condition

$$|f^{(m)}(x_1) - f^{(m)}(x_2)| = o(\sqrt{|x_1 - x_2|}) \qquad (-1 \leqslant x_1, x_2 \leqslant 1)$$

instead of (6).

Another interesting problem of this theory: what happens if we add the endpoints ± 1 to the nodes (1) which are the roots of orthogonal polynomials with support on $[-1, 1]$. Freud [12] proved that if $w(x) \geqslant m(1-x^2)^{1/2}$ $(m > 0)$, $\int_{-1}^{1} (w(x)/(1-x^2)^{1/2})\, dx < \infty$, and $f(x) \in \operatorname{lip} \frac{1}{2}$, then this extended Lagrange interpolation tends uniformly to $f(x)$ on $[-1, 1]$. In the conditions of this statement the weight function $w(x)$ is compared to the weight $(1-x^2)^{1/2}$ of the Chebyshev polynomials of second kind. This was generalized later by Vértesi [33] for Jacobi weights.

Freud was also interested in the mean convergence of Lagrange interpolation. Let $\alpha(x)$ be a real-valued nondecreasing bounded function defined for all real values of x, having infinitely many points of increase, and being uniquely defined by its moments $\int_{-\infty}^{\infty} x^n d\alpha(x)$ $(n=0, 1, 2,...)$. The Lebesgue–Stieltjes measure $d\alpha$ generated by $\alpha(x)$ is called "m-distribution." Let (1) be the roots of the orthogonal polynomials with respect to $d\alpha$. Generalizing earlier results of Erdös, Turán, and Shohat, Freud [5] proved that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} [f(x)-L_n(f, x)]^2 d\alpha(x)=0$$

provided $f(x)$ is of polynomial growth at $\pm\infty$, and $\int_{-\infty}^{\infty} f(x)^2 d\alpha(x)$ exists. Moreover, the condition about the polynomial growth can be replaced by a more general condition. (The mean convergence of Lagrange interpolation was later widely investigated by Nevai, see, e.g., [23].)

The significance of this result is that $d\alpha$ may have noncompact support. The investigation of Lagrange interpolation on infinite intervals raised new and unexpected difficulties, especially when one is concerned with pointwise convergence. Let (1) be the roots of the Hermite polynomials orthogonal to the weight-function e^{-x^2}. Then [13]

$$\begin{aligned}|f(x)-L_n(f, x)\leqslant K_r(\log n+e^{x^2/2})\, \omega_r(f; n^{-1/2})\\ +2\|f\|(2/e)^{n/2}e^{x^2/2} \qquad (-\infty<x<\infty),\end{aligned}$$

where $\omega_r\,(f\cdot)$ is the rth modulus of smoothness of the function $f(x)$, $\|f\|=\sup_{-\infty<x<\infty} |f(x)|$, and K_r depends only on r. Furthermore, this estimate is best possible, as for the order of convergence. These results were generalized later by Nevai [22] for Laguerre nodes, and by Kis [19] for the Markov–Sonin nodes.

Freud considered more general weights on $(-\infty, \infty)$ also. Let the weight-function $w(x)$ satisfy

$$w(x)\geqslant c_1 e^{-c_2x^2} \quad (-\infty<x<\infty), \qquad \int_{-\infty}^{\infty} e^{c_3x^2} w(x)\, dx<\infty$$

with some positive constants c_1, c_2, and c_3. Then [8] the weighted estimate

$$\lim_{n\to\infty} e^{-(1/2)c_2x^2}[f(x)-L_n(f, x)]=0$$

holds uniformly in $(-\infty, \infty)$ for the corresponding Lagrange interpolation, provided $f(x)$ is uniformly bounded and uniformly continuous on $(-\infty, \infty)$ and $f(x)\in \text{lip}\,\frac{1}{2}$ there. A similar result holds on $[0, \infty)$ for "Laguerre-type" weights.

If the nodes are on a finite interval, say on $[-1, 1]$, then of particular interest is the behavior of the Lagrange interpolation at the endpoints. Let the support of the m-distribution $d\alpha$ be $[-1, 1]$, let $p_n(x)$ be the corresponding orthonormal polynomials, and assume that

$$\int_{-1}^{1} (1-x)^{\rho-1} d\alpha(x) < \infty \qquad (0 < \rho \leqslant 1).$$

Then [14]

$$|f(1) - L_n(f, 1)| \leqslant \sqrt{P_{n-1}(1)\, P_n(1)} \cdot \varepsilon_n n^{-\rho},$$

where $\varepsilon_n = O(1)$ or $\varepsilon_n \to 0$ $(n \to \infty)$ provided $f(x) \in \operatorname{Lip} \rho$ or $f(x) \in \operatorname{lip} \rho$, respectively. (Similar result holds for $x = -1$.) This generalizes an earlier result of Kis established for the Chebyshev-nodes.

Freud extended the theory of strong approximation from Fourier series to interpolation. To state a special case of one of his results [11], let $d\alpha$ again be an m-distribution on $[-1, 1]$ such that (3) holds, let $P_n(x)$ be the corresponding orthonormal polynomials, $\alpha_n(x)$ a step-function with jumps $\lambda_n(x_{\nu n})$ at the nodes $x_{\nu n}$ $(\nu = 1, \dots, n)$ $(\lambda_n(x_{\nu n})$ are the Christoffel numbers). Further let

$$s_r^{(n)}(f, x) = \sum_{\nu=0}^{r} a_\nu^{(n)}(f)\, p_\nu(x)$$

$$\left(a_\nu^{(n)}(f) = \int_{-\infty}^{\infty} f(x)\, p_\nu(x)\, d\alpha_n(x),\ r \leqslant n-1 \right)$$

be the partial sums of the orthonormal "interpolating" expansion of $f(x)$ with respect to the distribution $d\alpha$. (The case $r = n-1$ leads to the Lagrange interpolation.) Then the following strong approximation type result holds:

$$\sum_{\nu=0}^{k} (\nu+1)^{s-1} |f(x) - s_\nu^{(n)}(f; x)|$$

$$\leqslant K(s, a, b) \sum_{\nu=0}^{k} (\nu+1)^{s-1} E_\nu(f) \qquad (x \in [a, b] \subset (-1, 1),\ s > 0,\ k < n),$$

where $E_\nu(f)$ is the best approximation of $f(x) \in C[-1, 1]$ by polynomials of degree at most n.

As for the $(C, 1)$ means of the Lagrange interpolation itself, it is known that they behave better than the original interpolating polynomials (although the contrast is not so sharp as in the case of Fourier series). In this connection Freud [10] proved that if the orthonormal polynomials $p_n(x)$ with respect to the m-distribution $d\alpha$ on $[-1, 1]$ satisfy

$$\sum_{\nu=0}^{n-1} p_\nu(x)^2 = O(n) \qquad (x \in M \subset [-1, 1]),$$

$\int_{-1}^{1} f(x)^2 d\alpha(x)$ exists and $f(x) \equiv 0$ $(x \in [a, b] \subseteq [-1, 1])$ then

$$\lim_{n \to \infty} \frac{1}{n} \sum_{\nu=0}^{n-1} |L_\nu(f, x) - f(x)| = 0 \qquad \left(x \in [a+\delta, b-\delta] \cap M, 0 < \delta < \frac{b-a}{2}\right),$$

where $L_\nu(f, x)$, as before, are the Lagrange interpolating polynomials based on the roots of $p_n(x)$. He gave estimate for the "strong Lebesgue function"

$$\Lambda_n(x) = \max_{|f(x)| \leqslant 1} \frac{1}{n} \sum_{\nu=1}^{n} |L_\nu(f, x)|$$

as well.

HERMITE–FEJÉR INTERPOLATION

While the Lagrange interpolation is never uniformly convergent for *all* continuous functions whatever the system of nodes (1) is, the situation is different for the so-called Hermite–Fejér interpolation. Let, as before, (1) be the roots of the polynomial $p_n(x)$ orthogonal with respect to the weight-function $w(x)$ with support in $[-1, 1]$. Let

$$h_{kn}(x) = \left[1 - \frac{p_n''(x_{kn})}{p_n'(x_{kn})}(x - x_{kn})\right] l_{kn}^2(x) \qquad (k = 1,..., n),$$

$$\tilde{h}_{kn}(x) = (x - x_{kn})\, l_{kn}^2(x) \qquad (k = 1, 2,..., n),$$

and

$$H_n(f, x) = \sum_{k=1}^{n} [h_{kn}(x) f(x_{kn}) + \tilde{h}_{kn}(x)\, d_{kn}],$$

where $f(x) \in C[-1, 1]$ and d_{kn} are certain numbers. It is well known that in case of the Jacobi polynomials (i.e., when $w(x) = (1-x)^\alpha (1+x)^\beta$, $\alpha, \beta > -1$) this process uniformly converges in $[-1, 1]$ or in every closed subinterval of $(-1, 1)$ according as $\max(\alpha, \beta) < 0$ or $\max(\alpha, \beta) \geqslant 0$, respectively, provided the d_{kn}'s satisfy certain growth condition. The proof of this statement makes heavy use of the differential equation for the Jacobi polynomials. The lack of known differential equations for a more general class of weight-functions makes it difficult to generalize the convergence theorem. Freud [2] succeeded in doing this. He proved that if $w(x) \in C[-1, 1]$, $w(x)$ satisfies the Dini–Lipschitz condition in $[a, b] \subset (-1, 1)$, the corresponding orthonormal polynomials satisfy (3) in $[a, b]$, and

$$d_{kn} = \begin{cases} o\left(\dfrac{n}{\log n}\right) & \text{if} \quad \alpha \leqslant x_{kn} \leqslant b, \\ o\left[\min\left(\dfrac{n}{\sqrt{1-x_{kn}^2}}, n^2\right)\right] & \text{for all} \quad x_{kn}, \end{cases}$$

then

$$\lim_{n \to \infty} H_n(f, x) = f(x) \tag{7}$$

for all $x \in (a, b)$ where $f(x)$ is continuous, provided $f(x)$ is bounded in $[-1, 1]$ and continuous at ± 1. If $f(x)$ is continuous in (a, b) then (7) holds uniformly in every closed subinterval of (a, b).

Moreover, if $w(x)$ satisfies (2) in $[-1, 1]$, $f(x) \in C[-1, 1]$, and $f(x)$ satisfies the Dini–Lipschitz condition

$$f(x_1) - f(x_2) = o\left(\log^{-1} \frac{1}{|x_1 - x_2|}\right) \qquad (a \leqslant x_1, x_2 \leqslant b)$$

then (7) holds uniformly in every closed subinterval of (a, b).

Another possibility for generalizing the convergence theorem for more general functions is the following: Assume that for a certain pair of numbers α, β, the functions $(1-x)^\alpha w(x)$ and $(1+x)^\beta w(x)$ are nondecreasing and nonincreasing, respectively. Then [16], for a bounded $f(x)$, if $f(x)$ is continuous at $x \in (-1, 1)$, (7) holds at this point. Further, if $f(x)$ is continuous in $[a, b] \subset [-1, 1)$ then the convergence in (7) is uniform in $[a, b]$.

In the opposite direction, (i.e., to what extent of conditions on $w(x)$ we can generalize the convergence theorem), Freud [15] proved that if for a fixed $\xi \in [-1, 1]$, the function $(x - \xi)\, w(x)$ is of bounded variation in $[-1, 1]$ then there exists an $f(x) \in C[-1, 1]$ such that (7) does not hold uniformly in $[-1, 1]$. (Here $d_{kn} = 0$, $k = 1, \ldots, n$.) In the special case $w(x) \equiv 1$, he raised the following problem: Is it true that if

$$f(1) = f(-1) = \int_{-1}^{1} f(x)\, dx, \tag{8}$$

then (7) holds uniformly in $[-1, 1]$? The necessity of (8) to the uniform convergence was shown earlier by Fejér. The answer for this problem was given by Schönhage [26], and later generalized by Szabados [27, 28, 29] and Vértesi [34].

LACUNARY INTERPOLATION

Though it was not in the center of his interest, Freud gave a remarkable contribution to the theory of lacunary interpolation investigated by Turán and his collaborators. In the simplest case of this theory, a polynomial

$$R_n(f, x) = \sum_{\nu=1}^{n} [f(x_{\nu n}) r_{\nu n}(x) + \beta_{\nu n} \rho_{\nu n}(x)] \tag{9}$$

is considered, which interpolates $f(x)$ at the nodes $x_{\nu n}$, and $\beta_{\nu n}$ are prescribed values for the *second* derivative. If the $x_{\nu n}$'s are the roots of the polynomial $(1-x^2) P_{n-1}^{(0,0)\prime}(x)$ and n is even, then (9) is uniquely determined. Generalizing a convergence theorem of Balázs and Turán, he proved [4] that if $\omega_2(f, h) = o(h)$,

$$\beta_{\nu n} = o\left(\frac{n}{\sqrt{1-x_{\nu n}^2}}\right) \quad (\nu = 2, ..., n-1),$$

$$\beta_{0n} = o(n^2), \qquad \beta_{nn} = o(n^2);$$

then

$$\lim_{n \to \infty} R_n(f, x) = f(x)$$

uniformly in $[-1, 1]$. A quantitative version of this statement was proved later by Vértesi [31].

APPLICATIONS

Interpolation procedures become more useful for practical purposes if we loosen the strict condition of Lagrange interpolation, and permit polynomials, of degree cn $(c > 1)$ to interpolate at the nodes (1). In fact, one can obtain the direct theorems of the theory of best approximation this way. At the Oberwolfach conference in 1963, Butzer raised the problem whether there are interpolating polynomials directly proving the Jackson theorem. It was Freud [6] who answered first this question by constructing polynomials of degree $4n-3$ interpolating at $(n/3) + O(1)$ nodes and approximating in the order of $\omega(f, 1/n)$. Later, in a joint paper with

Vértesi [9] they slightly modified the construction and obtained a polynomial operator $J_n^*(f, x)$ of degree at most $4n-2$ which interpolates at the Chebyshev nodes $\cos((2k-1)/2n)\,\pi$ $(k=1,\dots,n)$, and provide a Timan type estimate

$$|f(x)-J_n^*(f,x)| \leqslant C\left[\omega\left(f, \frac{\sqrt{1-x^2}}{n}\right)+\omega\left(f, \frac{1}{n^2}\right)\right] \qquad (|x| \leqslant 1).$$

Later, this result was extended to the construction of operators based on general Jacobi nodes, in a joint work with Sharma [17, 18]. Here they were able to decrease the degree of the operator to $n(1+\varepsilon)$, $\varepsilon>0$ arbitrary.

His original work [6] started a flow of investigations of this type; to name only a few, we mention related works of Kis, Vértesi [20, 30, 32], Sallay [24], Saxena [25], and Mathur [21]. Saxena [25] was the first to obtain the even stronger estimate $C\omega(\sqrt{1-x^2}/n)$ by interpolation (the famous result of Teljakowski–Gopengauz).

It is a classical result of Bernstein that to a suitable system of nodes (1) on $[-1, 1]$ and any $\varepsilon>0$, there exists a sequence of linear operators $L_n(f, x)$ with the following properties: for any $f(x) \in C[-1, 1]$,

(a) $L_n(f, x)$ is of degree at most $n(1+\varepsilon)$;

(b) $L_n(f, x_{kn})=f(x_{kn})$ $(k=1,\dots,n)$;

(c) $\lim_{n\to\infty} L_n(f, x)=f(x)$.

Erdös gave a necesary and sufficient condition for a system of nodes (1) to have this property: these systems are called "Bernstein–Erdös-type." Let us call a system (1) "well approximating", if (a) and (b) hold, and (c) is replaced by the stronger condition

(c*) $|f(x)-L_n(f, x)| \leqslant K(\varepsilon)\, E_n(f) \quad (|x| \leqslant 1)$,

where $K(\varepsilon)$ depends only on ε and the system (1). Freud [7] proved the following interesting connection between these two characterizations: A system (1) is well approximating if and only if it is of Bernstein–Erdős-type.

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Printed in Belgium